Journal of Statistical Physics, Vol. 118, Nos. 1/2, January 2005 (© 2005)
DOI: 10.1007/s10955-004-8784-6

Existence and Uniqueness of Solutions for the Couette Problem

S. Ghomeshi[1,2]

Received July 5, 2003; accepted August 25, 2004

We study existence and uniqueness results for the one-dimensional Boltzmann equation with inflow and diffusive boundary conditions. Our focus, partly encompasses some of the properties of the Boltzmann collision gain term which play a significant role in existence and uniqueness results. A series of estimates are proven on the collision term which is shown to produce a suitable function space in which the contraction mapping arguments are available.

KEY WORDS: Existence; uniqueness; collision term; estimates; contraction.

1. INTRODUCTION

Throughout this paper we let $v=(v_1, v_2, v_3)\in\mathbb{R}^3$ denote a velocity vector with $x-$, $y-$ and z components v_1, v_2, v_3. In addition, we use x to denote the (one-dimensional) position variable in the interval $[0, 1]$. We study the one-dimensional Boltzmann equation in the slab given by

$$v_1\frac{\partial f}{\partial x}=J(f, f), \tag{1}$$

where one recalls $f=f(x, v)$ to be the one particle distribution function and $J(f, f)$ the collision operator describing the rate of change of f due

[1]Department of Mechanical and Industrial Engineering, Concordia University, Room H-549, 1455 de Maisonneuve Blvd. West Montreal, Quebec, Canada H3G 1M8; e-mail: shahin@mie.utoronto.ca
[2]This work was done as part of the author's M.Sc. studies at the University of Victoria.

to binary collisions between gas particles. The bilinear operator $J(f, f)$ takes on the explicit form

$$J(f,f)=\int_{\mathbb{R}^3\times\mathbb{S}^2}[f(v')f(v'_*)-f(v)f(v_*)]B(v-v_*,n)\,dn\,dv_*,$$

where n is a unit vector in the unit sphere $\mathbb{S}^2\{n\in\mathbb{R}^3\,\big|\,|n|=1\}$ with Lebesgue measure dn. $v, v_*\in\mathbb{R}^3$ are the pre-collisional velocities and v', v'_*, the post-collisional velocities. In the event of a collision, (v, v_*) will undergo the transformation

$$(v,v_*)\to(v',v'_*),$$

where the post-collisional velocities are expressed in terms of the pre-collisional velocities by the relations

$$\begin{aligned}v'&=v-n(n\cdot(v-v_*)),\\ v'_*&=v_*+n(n\cdot(v-v_*)).\end{aligned}$$

B is the collision kernel and it depends on $|v-v_*|$ and $(n\cdot(v-v_*))$ only. In general, after an angular cutoff it takes the form

$$B(v-v_*)=|v-v_*|^{\beta}h(\theta),$$

where θ is the polar angle of n relative to a polar axis in direction $v-v_*$, and h is assumed to be an integrable function on $[0,\pi]$ with $\int_{s_+}h(\theta)\,dn=1$, where s_+ is the hemi-sphere corresponding to $(v-v_*,n)>0$. The integer β is chosen from the set $\{-1,0,1\}$, which describes Maxwellian molecules ($\beta=0$), a hard sphere gas ($\beta=1$), and a soft sphere gas ($\beta=-1$). According to Grad[14] B satisfies the following conditions:

- $B\in L^{\infty}_{loc}(\mathbb{R}^3,\mathbb{S}^2)$,
- $B(v,n)\leqslant b_1\frac{|(n\cdot v)|}{|v|}(1+|v|^{\gamma})$,
- $\int_{\mathbb{S}^2}B(v,n)\,dn\geqslant b_{\circ}|v|(1+|v|)^{-1}$,

where $\gamma\in[0,1]$ and $b_{\circ}$, b_1 are positive constants. Although generalizations are available, we will deal with the case, where we have only hard sphere interactions, where B yields the expression

$$B(v-v_*,n)=|n\cdot(v-v_*)|=|(v-v_*)||\cos\theta|.$$

The bilinear operator $J(f, f)$ is usually decomposed as

$$J(f, f) = J^+(f, f) - J^-(f, f),$$

where $J^+(f, f)$ is the gain term and $J^-(f, f)$ is the loss term due to binary collisions of gas particles. It is immediate that this decomposition yields the following expressions:

$$\begin{aligned} J^+(f, f) &= \int_{\mathbb{R}^3 \times \mathbb{S}^2} f(v') f(v'_*) B(v - v_*, n)\, dn\, dv_*, \\ J^-(f, f) &= f \nu(f), \\ \nu(f) &= \int_{\mathbb{R}^3_v} f(v_*) \left[\int_{\mathbb{S}^2} B(v - v_*, n)\, dn \right] dv_*, \end{aligned}$$

where $\nu(f)$ determines the frequency of collisions associated with the distribution function f.

Arkeryd *et al.*[1] have studied the global existence problem for the steady Boltzmann equation in a slab with inflow boundary conditions. They introduce a measure theoretic formulation of the problem, and for a truncated collision kernel, they prove existence of a solution based on weak* compactness of uniformly bounded measures. Arkeryd and Nouri[2] solve the problem with both inflow and diffuse reflective boundary conditions. However, their arguments rely on using the boundedness of the Boltzmann entropy production together with the boundedness of mass and energy, in order to prove weak compactness in L^1. They then solve the existence problem for a sequence of functions for which these estimates hold. Illner *et al.*,[15] have generalized the work in ref. 1 and prove global existence for the case of purely diffusive boundary conditions. They have also obtained both existence and uniqueness of solutions for one, two, and three dimensions provided that the size of the domain is made sufficiently small.

One common shortcoming of all these earlier studies is the unphysical truncations, which are made on the collision kernel. The reasons for these truncations are twofold: first, in a neighborhood of zero in the velocity space, the non-linear term in the steady Boltzmann equation becomes significantly large. Therefore, the kernel is truncated in such a way that collisions between particles having small velocities are ignored. Second, for hard sphere interactions, the collision term becomes unbounded as $v \to \infty$, and so additional truncations are made in order to control the magnitude of the velocities so that it will not grow without bound.

In more recent work, general L^1 solutions for the stationary, full non-linear equations of Boltzmann type have been obtained by weak compactness techniques, under no other restrictions except Grad's angular cut-off. Examples include the Povzner equation in bounded domains of $\mathbb{R}^n$, as obtained in ref. 3 and general L^1 solutions of the stationary non-linear Boltzmann equation in a slab.(4) In these papers the entropy dissipation term provides the most useful control and is the tool used to get rid of the small velocity truncations.

The slab case was the beginning of a long series of papers, where the stationary Boltzmann equation was studied in a Couette setting between two coaxial, rotating cylinders (two-roll configuration) with given Maxwellian indata on the cylinders as in refs. 6–9. The, first three papers of the series, focused on the close to equilibrium frame. However, further improvements are made in ref. 9, where the existence results far from equilibrium for the two-roll problem are studied. Here, the use of the entropy dissipation control in delivering existence results has been generalized from the slab case, to cylinders. A particular $\mathbb{R}^2$ case for the two-roll configuration is studied, where existence is shown without any small velocity truncations. This is a big improvement since up until now, the removal of the small velocity cut-off for the nonlinear, stationary Boltzmann equation with large boundary data, remained an open problem in more than one space dimensions. We refer the reader to a previous study by Arkeryd and Nouri,(5) where existence results for the n-dimensional case, could be obtained only under supplementary small velocity truncations.

Although, the more recent studies have advanced in proving existence results for the non-linear stationary Boltzmann equation without the small velocity truncations, they do not yield uniqueness being that they are essentially based on compactness arguments. The focus of this paper is on proving both existence and uniqueness of solutions to the one-dimensional steady Boltzmann equation with inflow and diffusive reflective boundary conditions without truncations on the collision kernel. This was established for large mean free paths by Maslova.(16) In her work, it is shown that the collision operator has special properties which make it possible to avoid truncations. These properties are entailed in estimates on the collision term which are then needed in producing function spaces with the contractive property.

The results presented, are based on sketches provided in Maslova's monograph,(16) where the proofs are either left out or incomplete. Here, we present the ideas clearly, and provide rigorous proofs of the Lemma's, which lead to the main theorem. Although, the contribution of the current paper is mostly in filling in details and adding clarity to the work of Maslova, some modifications have also been made. Lemma 2.5 in Chapter 3 of Maslova's monograph[16] has been modified, and the restriction to convex functions removed. This lemma, which is presented as Lemma 2 in the current paper, establishes some regularity in the velocity space. This is a key lemma in providing the necessary bounds on the collision term.

The scope of this paper is as follows: in Section 2, we study the steady Boltzmann equation with inflow boundary conditions, where we reformulate the problem as a fixed point problem. In Section 3, certain estimates are proved, revealing the special nature of the collision term. These estimates are then used, in addition to the contraction mapping theorem in proving existence and uniqueness for the inflow case in Section 4. The rest of the paper is dedicated to the diffuse reflective case. We make one final remark. The boundary value problems discussed here are restricted to particles in small bounded domains, and hence are not global in nature as in the Arkeryd and Nouri papers. However, the results are constructive and prove both existence and uniqueness.

2. INFLOW BOUNDARY CONDITIONS

In the rest of this paper, we will deal with existence and uniqueness of solutions to the steady Boltzmann equation in a slab with given boundary conditions. In particular in this section we will deal with the boundary value problem

$$v_1 \frac{\partial f}{\partial x} = \varepsilon J(f, f), \quad x \in (0, 1), \tag{2}$$

$$f(0, v) = f^-(0, v), \quad v_1 > 0, \tag{3}$$

$$f(1, v) = f^-(1, v), \quad v_1 < 0 \tag{4}$$

in which the size of the domain is bounded in terms of a small parameter ε; which is a measure of the inverse mean free path of a particle. The boundary conditions (3) and (4) are called inflow boundary conditions, where a function $f^-(i, v), i = \{0, 1\}$ is prescribed at the boundary, i.e., the two end points of the slab. We proceed with a representation of the solution to problem (2)–(4) into an integral equation, which is later shown to have a unique fixed point.

2.1. Steady Solution Operators

From the decomposition: $J(f, f) = J^+(f, f) - f\nu(f)$, Eq. (2) is written as

$$\frac{\partial f(x, v)}{\partial x} + \frac{\varepsilon}{v_1} f\nu(f) = J^+(f, f).$$

Holding v fixed, we treat the equation as an ordinary differential equation. Therefore, multiplying through by the integrating factor

$$\exp\{\varepsilon v_1^{-1} \int_y^x \nu(z, v)\, dz\}$$

and applying the boundary conditions (3) and (4) we convert the problem to the integral equation

$$f = A(f), \quad A(f) = Wf^- + \varepsilon U J^+(f, f), \tag{5}$$

where the operators W, U admit the following representations:

$$(Wf^-)(x, v) = f^-(\chi(v_1), v)\Pi(\chi(v_1)), \tag{6}$$

$$(UJ^+(f, f))(x, v) = v_1^{-1} \int_{\chi(v_1)}^x J^+(f, f)(y, v)\Pi(x, y)\, dy \tag{7}$$

with

$$\chi(v_1) = \frac{1}{2}(1 - \text{sgn } v_1),$$
$$\Pi(x, y) = \exp\left\{-\varepsilon v_1^{-1} \int_y^x \nu(z, v)\, dz\right\},$$
$$\nu(x, v) = \nu(f)(x, v).$$

Remark 1. We notice that even though v_1 can be positive or negative, the sign of the operator $UJ^+(f, f)(x, v)$ will always be non-negative. This is an important observation for the proof of Lemma 6.

2.2. Norms Defined

We now introduce the weight function

$$\varphi(v) = \exp(s|v|^2)(1 + |v|^2)^{r/2}, \quad s \geqslant 0, \quad r \geqslant 0 \tag{8}$$

with the weighted norms

$$\|f\|_{-1} = \sup_{\omega} \int_{\mathbb{R}_v^3} \varphi(v)|v - \omega|^{-1} \|f(., v)\|_{L^\infty}\, dv, \tag{9}$$

$$\|f\| = \int_{\mathbb{R}_v^3} \varphi(v)\|f(., v)\|_{L^\infty}\, dv, \tag{10}$$

$$\|f\|_2 = \sup_E \int_E \varphi(v)\|f(., v)\|_{L^\infty(0,1)}\, d\sigma(v), \tag{11}$$

where the sup in (11) is to be taken over all planes E in $\mathbb{R}_v^3$. In addition, we consider the Banach space

$$X_\circ = \{f : \varphi f \in L^1(\mathbb{R}_v^3, L^\infty(\Omega))\} \tag{12}$$

with the norm $\|.\|$ defined above. We will use the contraction mapping theorem to prove that Eq. (5) has a unique solution for a sufficiently small ε. However, first we must define the proper function set in which the contraction mapping arguments are available. Henceforth, we let $a_j (j = 1, 2, 3)$ be some positive constants and consider sets $\mathcal{A} \subset X_\circ$ defined by

$$\mathcal{A} = \{f \in X_\circ : f \geqslant 0, \|f\| \leqslant a_1, \nu(f) \geqslant a_2, \|f\|_{-1} \leqslant a_3, \|f\|_2 \leqslant a_4\}.$$

We will show that there are constants $\varepsilon, a_1, a_2, a_3$, and a_4 such that the Banach fixed point theorem is applicable in $\mathcal{A}$. The following Lemma will then give the necessary contraction property and will be proved in Section 4.

Lemma 1. There exists a constant C such that

$$\|Af - Ag\| \leqslant C\varepsilon \ln \frac{1}{\varepsilon} \|f - g\|, \quad s > 0, \ \ r \geqslant 1$$

for $f, g \in \mathcal{A}$.

Upon the proof of Lemma 1, for sufficiently small ε we can state the main result of Section 4, namely.

Theorem 1. There exists a constant a_5 such that the problems (2)–(4), has a unique solution if $\varepsilon \leqslant a_5$.

The proof of the key contraction bound depends on certain estimates on the Boltzmann collision operator. These estimates exploit the properties of the collision gain term, which make it possible to prove Theorem 1 without unphysical truncations on the collision kernel. These estimates are discussed further in Section 3.

3. SOME IMPORTANT ESTIMATES

We begin with a lemma, which shows that the collision operators introduce some regularity in the velocity space. The proof of this lemma is used later in establishing other bounds on the collision operator. As we will see in the following sections, these estimates are used in proving the required existence and uniqueness results.

3.1. A Regularity Estimate

Lemma 2. Let

$$\hat{V}_k(f) = \sup_{\omega} \int_{\mathbb{R}^3_v} |\omega - p|^{-k} f(p)\, dp \quad \text{for} \quad 0 < k < 2.$$

Then for $\gamma \in [0, 1]$ and any positive function h satisfying $h(q) \leqslant h(v)h(v_*)$, where $q = v - n(n \cdot (v - v_*))$, we have

$$\hat{V}_k(hJ^+(f, g)) \leqslant C(2-k)^{-1} \hat{V}_\circ(fh) \hat{V}_{k-\gamma}(gh)$$

for some positive constant C.

Remark 2. In order to fully clarify the statement of Lemma 2, we give some examples of the type of functions which satisfy the inequality $h(q) \leqslant h(v)h(v_*)$. For later purposes we look at functions of the type

$$h_1(q) = (1 + |q|^2)^{r/2} \quad \text{and} \quad h_2(q) = \exp(s|q|^2)$$

for $r \geqslant 0, s > 0$. We know that for particle velocities q, q_*, v, v_*, lying on the collision sphere, the conservation of energy dictates that $|q|^2 + |q_*|^2 = |v|^2 + |v_*|^2$, where we recall that v, v_* are the pre-collisional velocities and q, q_* the post-collisional velocities. Therefore, it is easily verified that both $h_1(q)$ and $h_2(q)$ satisfy the requirements in Lemma 2.

Proof. Let $\bar{\varphi}(v) = h(v)|\omega - v|^{-k}$, and consider the average

$$\langle \bar{\varphi}(v) J^+(f, g) \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} f(v) g(v_*) \bar{\varphi}(v') B(v - v_*, n)\, dv\, dv_*\, dn. \quad (13)$$

We make the change of variables $n \to q = v - n(n \cdot (v - v_*))$, where n is a unit vector which lives in the unit sphere $\mathbb{S}^2$ directed along the vector $v - v'$. q is a vector in $\mathbb{R}^3$ which lives in the sphere K_{vv_*} defined by

$$K_{vv_*} = \left\{ q \in \mathbb{R}^3 \,\middle|\, |q - \tfrac{1}{2}(v + v_*)| = \tfrac{1}{2}|v - v_*| \right\}.$$

We transform the integration variable n on the unit sphere to a new sphere K_{vv_*} of radius $1/2|v - v_*|$.

From the definition of the set K_{vv_*} we notice that q can also be written as

$$q = \frac{1}{2}(v + v_*) + \frac{1}{2}|v - v_*|e, \quad e \in \mathbb{S}^2_*,$$

where $\mathbb{S}^2_*$ is the sphere with normal vector e having components

$$e = (\sin\phi_1 \cos\theta_1, \sin\phi_1 \sin\theta_1, \cos\phi_1).$$

We want to express the polar and azimuthal angles of the unit sphere, in terms of the polar and azimuthal angles of the new sphere with unit normal e. Now choosing the polar axis in direction $v - v_*$, i.e., $n \cdot (v - v_*) = |v - v_*| \cos\phi$, we have the following relations:

$$\begin{aligned}
q_1 &= v_1 - \sin\phi \cos\theta \cos\phi |v - v_*| &&= \tfrac{1}{2}(v_1 + v_{*1}) + \tfrac{1}{2}|v - v_*| \sin\phi_1 \cos\theta_1,\\
q_2 &= v_2 - \sin\phi \sin\theta \cos\phi |v - v_*| &&= \tfrac{1}{2}(v_2 + v_{*2}) + \tfrac{1}{2}|v - v_*| \sin\phi_1 \sin\theta_1,\\
q_3 &= v_3 - \cos^2\phi |v - v_*| &&= \tfrac{1}{2}(v_3 + v_{*3}) + \tfrac{1}{2}|v - v_*| \cos\phi_1,
\end{aligned}$$

where $v = (v_1, v_2, v_3)$ and $v_* = (v_{*1}, v_{*2}, v_{*3})$. Choosing Cartesian coordinates (consistent with polar axis in direction $v - v_*$) such that

$$v - v_* = \begin{pmatrix} 0 \\ 0 \\ v_3 - v_{*3} \end{pmatrix},$$

we arrive at the relations,

$$\phi_1 = 2\phi - \pi \quad \text{and} \quad \theta_1 = \pi + \theta.$$

Therefore, we have,

$$\sin\phi = \frac{\sin\phi_1}{2\cos\phi} \quad \text{and} \quad \cos\phi = \frac{(n \cdot (v - v_*))}{|v - v_*|} = \frac{|q - v|}{|v - v_*|}$$

in which we can then express dn in terms of dq through the relation,

$$\begin{aligned}
\sin\phi \, d\phi \, d\theta &= \frac{\sin\phi_1}{2\cos\phi} \left(\frac{1}{2} d\phi_1\right) d\theta_1\\
&= \frac{1}{|q - v||v - v_*|} \underbrace{\frac{1}{4}|v - v_*|^2 \sin\phi_1 \, d\phi_1 \, d\theta_1}_{dq}\\
&= \frac{1}{|q - v||v - v_*|} dq.
\end{aligned}$$

Summarizing,

$$\langle \bar{\varphi}(v) J^{+}(f,g)\rangle = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v) g(v_*) |v-v_*|^{-1} I \, dv \, dv_*, \tag{14}$$

where

$$I = \int_{K_{vv_*}} \bar{\varphi}(q) B |q-v|^{-1} \, dq.$$

With respect to the kernel B, it is sufficient to assume that B satisfies

$$B(v,n) \le b_1 \frac{|n.v|}{|v|} |v|^{\gamma}$$

with $\gamma \in [0, 1]$, $b_1 > 0$ consistent with Grad's assumptions mentioned in the introduction. Thus, we have that for $q \in K_{vv_*}$, $B(v-v_*, n)$ satisfies the estimate

$$|q-v|^{-1} B \leqslant b_1 |v-v_*|^{\gamma-1}.$$

From the definition of $\hat{V}_k(f)$, the above estimate on B, and the fact that $\bar{\varphi}(q) = h(q)|\omega - q|^{-k}$ we write

$$\begin{aligned}\hat{V}_k(hJ^{+}(f,g)) &= \sup_{\omega} \int_{\mathbb{R}^3_v} |\omega - q|^{-k} h(q) J^{+}(f,g) \, dq \\ &\leqslant \sup_{\omega} b_1 \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v) g(v_*) |v-v_*|^{\gamma-2} I_1 \, dv \, dv_*,\end{aligned}$$

where

$$I_1 = \int_{K_{vv_*}} h(q) |\omega - q|^{-k} \, dq.$$

Since $h(q) \leqslant h(v)h(v_*)$, I_1 is estimated as

$$I_1 \leqslant h(v) h(v_*) I_2$$

with

$$I_2 = \int_{K_{vv_*}} |\omega - q|^{-k} \, dq.$$

Hence we are only left with estimating the integral I_2. Taking

$$\omega = |\omega| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and changing into polar coordinates, we have

$$\begin{aligned} |\omega - q| &= |R\sin\phi\cos\theta, R\sin\phi\sin\theta, |\omega| - R\cos\phi| \\ &= \left[R^2 - 2R|\omega|\cos\phi + |\omega|^2\right]^{1/2}. \end{aligned}$$

Thus the integral I_2 is written as

$$I_2 = 2\pi R^2 \int_0^\pi (R^2 - 2R|\omega|\cos\phi + |\omega|^2)^{-k/2} \sin\phi \, d\phi,$$

where $R = (1/4)|v - v_*|$. Integrating, we have

$$I_2 = \frac{2\pi}{2-k} R^{2-k} \frac{\left[(1 + (|\omega|/R))^{2-k} - (1 - (|\omega|/R))^{2-k}\right]}{|\omega|/R} \quad \text{for} \quad |\omega| \leqslant R$$

and

$$I_2 = \frac{2\pi}{2-k} R^{2-k} \frac{\left[((|\omega|/R) + 1)^{2-k} - ((|\omega|/R) - 1)^{2-k}\right]}{|\omega|/R} \quad \text{for} \quad |\omega| > R.$$

Letting $x = |\omega|/R$, we make the observation that for $0 \le k < 2$,

$$\begin{aligned} \frac{(1+x)^{2-k} - (1-x)^{2-k}}{x} &\leqslant \frac{(1+x)^2 - (1-x)^2}{x} = 4, \quad x \le 1 \\ \frac{(x+1)^{2-k} - (x-1)^{2-k}}{x} &\leqslant \frac{(x+1)^2 - (x-1)^2}{x} = 4, \quad x > 1. \end{aligned}$$

Therefore,

$$I_2 \le 8\pi(2-k)^{-1} R^{2-k}, \tag{15}$$

which yields the estimate,

$$\begin{aligned} I_1 &\leqslant h(v)h(v_*)I_2 \\ &\leqslant Ch(v)h(v_*)(2-k)^{-1}|v-v_*|^{2-k} \end{aligned}$$

for some positive constant C. Finally,

$$\begin{aligned} &\hat{V}_k(hJ^+(f,g)) \\ &\leqslant C(2-k)^{-1}\sup_{\omega}\int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_*)|v-v_*|^{\gamma-k}h(v)h(v_*)\,dv\,dv_* \\ &= C(2-k)^{-1}\sup_{\omega}\int_{\mathbb{R}^3} f(v)h(v)\,dv\int_{\mathbb{R}^3} g(v_*)h(v_*)|v-v_*|^{\gamma-k}\,dv_* \\ &= C(2-k)^{-1}\hat{V}_\circ(fh)\hat{V}_{k-\gamma}(gh). \quad \blacksquare \end{aligned}$$

3.2. An Estimate on $\|J^+(f,f)\|_{-1}$

Lemma 3. There exists a positive constant C such that

$$\|J^+(f,g)\|_{-1} \leqslant C\|f\|\,\|g\|$$

for all $f, g \in L^\infty$.

Proof. We showed in Lemma 2 that functions of the form $\varphi(v)$ satisfy the condition $\varphi(q) \leqslant \varphi(v)\varphi(v_*)$ for velocities q, v, v_* lying on the collision sphere. Hence, we can apply Lemma 2 with $k=1, \gamma=1$ and get the estimate

$$\hat{V}_1(\varphi J^+(f,g)) \leqslant C\hat{V}_\circ(\varphi f)\hat{V}_\circ(\varphi g).$$

Applying the definition of $\hat{V}_k$, we have

$$\begin{aligned} \hat{V}_\circ(\varphi f) &= \int_{\mathbb{R}^3_v} \varphi(v)f(v)\,dv \leqslant \|f\|, \\ \hat{V}_\circ(\varphi g) &= \int_{\mathbb{R}^3_v} \varphi(v)g(v)\,dv \leqslant \|g\|, \end{aligned}$$

which leads to the result

$$\hat{V}_1(\varphi J^+(f,g)) \leqslant C\|f\|\,\|g\|.$$

Therefore, adapting the definition of $\|.\|_{-1}$, and observing that

$$\sup_x J^+(f,g)(v) \leqslant J^+\left(\sup_x f(x,v), \sup_x g(x,v_*)\right)(v)$$

the main result follows:

$$\begin{aligned}\|J^+(f,g)\|_{-1} &\leqslant \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1} J^+(\sup_x f, \sup_x g)(v)\,dv\\ &= \hat{V}_1(\varphi J^+(\sup_x f, \sup_x g))\\ &\leqslant C\|f\|\,\|g\|. \quad \blacksquare\end{aligned}$$

3.3. An Estimate on $\|J^+(f,f)\|$

Lemma 4. For the norms $\|.\|_{-1}$, and $\|.\|$ defined in (9) and (10), we have

$$\|J^+(f,f)\| \leqslant \pi s^{-1}\|f\|_{-1}\|f\|, \quad s>0.$$

Proof. In order to estimate $\|J^+(f,g)\|$ we first evaluate the integral

$$\int_{K_{vv*}} \exp\{s|q|^2\}\,d\sigma(q)$$

in which K_{vv_*} was defined in Section 3.1, with a unit vector e, given by the components

$$e=(\sin\phi_1\cos\theta_1, \sin\phi_1\sin\theta_1, \cos\phi_1)$$

and where it was also shown that

$$q=\frac{1}{2}(v+v_*)+\frac{1}{2}|v-v_*|e \quad \text{and} \quad d\sigma(q)=\frac{1}{4}|v-v_*|^2\sin\phi\,d\phi\,d\theta.$$

Let $r=(1/2)|v-v_*|$ be the radius of the sphere K_{vv_*}. Then

$$\begin{aligned}&\int_{K_{vv*}} \exp\{s|q|^2\}\,d\sigma(q)\\ &=r^2\int_0^\pi\int_0^{2\pi} e^{s(v+v_*)^2/4+sr(v+v_*)\cdot e+s|v-v_*|^2/4}\sin\phi_1\,d\phi_1\,d\theta_1\\ &=2\pi r^2 e^{s(v+v_*)^2/4}\int_0^\pi e^{sr(v+v_*)\cdot e+s|v-v_*|^2/4}\sin\phi_1\,d\phi_1.\end{aligned}$$

If we take the polar direction as $v+v_*$, then

$$r(v+v_*).e = |v+v_*| r \cos\phi_1$$

and

$$\int_{K_{vv*}} \exp\{s|q|^2\}\, d\sigma(q) \\ = \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} \left[e^{s(|v+v_*|+|v-v_*|)^2/4} - e^{s(|v+v_*|-|v-v_*|)^2/4} \right].$$

Now we let $\varphi(q) = \exp\{s|q|^2\}$, and for q, v, v_* lying on the collision sphere, $\varphi(q) \leqslant \varphi(v)\varphi(v_*)$. Then, we obtain the estimate

$$\int_{K_{vv*}} \varphi(q)\, d\sigma(q) \leqslant \pi s^{-1} \frac{|v-v_*|}{|v+v_*|} \varphi(v)\varphi(v_*). \tag{16}$$

From Lemma 2, we have for any function $\varphi(v) \in \mathrm{C}(\mathbb{R}^3_v)$, $\varphi \geqslant 0$, that

$$\langle \varphi J^+(f,f) \rangle \\ = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(v) f(v_*) |v-v_*|^{-1} \left\{ \int_{K_{vv*}} \varphi(q) B |q-v|^{-1}\, d\sigma(q) \right\} dv\, dv_*.$$

For hard sphere collisions, where $B = |q-v||\cos\theta| \leqslant |q-v|$,

$$\langle \varphi(v) \sup_x J^+(f,f) \rangle \\ \leqslant \int_{\mathbb{R}^3 \times \mathbb{R}^3} \sup_x f(v) \sup_x f(v_*) |v-v_*|^{-1} \left\{ \int_{K_{vv*}} \varphi(q)\, d\sigma \right\} dv\, dv_*.$$

Applying the estimate in (16), and using the norms defined in (9) and (10) we obtain

$$
\begin{aligned}
&\|J^+(f,f)\| \\
&\leqslant \pi s^{-1} \int_{\mathbb{R}^3\times\mathbb{R}^3} \sup_x f(v) \sup_x f(v_*) |v-v_*|^{-1} \frac{|v-v_*|}{|v+v_*|} \varphi(v)\varphi(v_*)\, dv\, dv_* \\
&\leqslant \pi s^{-1} \sup_v \int_{\mathbb{R}^3} \varphi(v) \sup_x f(v)\, dv \left\{ \int_{\mathbb{R}^3} \varphi(v_*) |v+v_*|^{-1} \sup_x f(v_*)\, dv_* \right\} \\
&= \pi s^{-1} \sup_v \int_{\mathbb{R}^3} \varphi(v) \|f(.,v)\|_{L^\infty(0,1)}\, dv \\
&\quad \times \left\{ \int_{\mathbb{R}^3} \varphi(v_*) |v+v_*|^{-1} \|f(.,v_*)\|_{L^\infty(0,1)}\, dv_* \right\} \\
&= \pi s^{-1} \|f\|_{-1} \|f\|. \quad \blacksquare
\end{aligned}
$$

3.4. An Estimate on $\|J^+(f,f)\|_2$

Lemma 5. There exists a positive constant C_1 independent of f such that

$$\|J^+(f,f)\|_2 \leqslant C_1 \|f\|^2 \quad s \geqslant 0, r \geqslant 0,$$

where s and r are the parameters in the weight function $\varphi(v)$, and $\|.\|_2$ is the norm defined in (11).

Proof. We define a weight function $\varphi_\alpha(v)$ in such a way so that it will concentrate on a plane E in $\mathbb{R}^3$ as $\alpha \to \infty$. Without restricting the generality we assume E to be the xy-plane. With this in mind, we choose $\varphi_\alpha(v)$, to be

$$\varphi_\alpha(v) = \varphi(v) \left(\frac{\alpha}{\pi}\right)^{1/2} \exp(-\alpha v_3^2),$$

where $\varphi(v)$ is defined in Section 2. By Lemma 2, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3_v} \varphi_\alpha(v) J^+(f,f)\, dv \qquad (17) \\
&\quad = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(v_*) \left\{ \int_{K_{vv_*}} \varphi_\alpha(q) \frac{B(v-v_*, n)}{|q-v||v-v_*|}\, d\sigma(q) \right\} dv\, dv_*.
\end{aligned}
$$

We take $|v_3|$ to be the distance of v from the xy-plane. We see by taking the limit $\alpha \to \infty$, that the only contribution of $\varphi_\alpha(v)$ is when $v_3 = 0$, hence concentrating on the plane E. Now we focus on the inner integral (in brackets) of Eq. (18), which we abbreviate by I. Since $|B| \leqslant b_1 |q-v||v-v_*|^{\gamma-1}$, we have

$$\begin{aligned} I &\leqslant b_1 \int_{K_{vv_*}} \varphi_\alpha(q)|v-v_*|^{\gamma-2}\, d\sigma(q) \\ &= b_1|v-v_*|^{\gamma-2} \int_{K_{vv_*}} \varphi_\alpha(q)\, d\sigma(q) \\ &= b_1|v-v_*|^{\gamma-2} \int_{K_{vv_*}} \varphi(q) \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha q_3^2}\, d\sigma(q). \end{aligned}$$

From Lemma 2, we have

$$I \leqslant b_1|v-v_*|^{\gamma-2} \left(\frac{\alpha}{\pi}\right)^{1/2} \varphi(v)\varphi(v_*) \int_{K_{vv_*}} e^{-\alpha q_3^2}\, d\sigma(q)$$

so now we need to estimate the integral

$$\int_{K_{vv_*}} e^{-\alpha q_3^2}\, d\sigma(q).$$

To do this, we exploit the geometry of the problem. We consider the case where K_{vv_*} does not intersect the plane E, (as seen in Fig. 1) and that its origin is situated at the center of the sphere. Therefore, we write

$$\int_{K_{vv_*}} e^{-\alpha q_3^2}\, d\sigma(q) = \frac{|v-v_*|^2}{4} \int_0^{2\pi} \int_0^{\pi} e^{-\alpha q_3^2} \sin\phi\, d\phi d\theta,$$

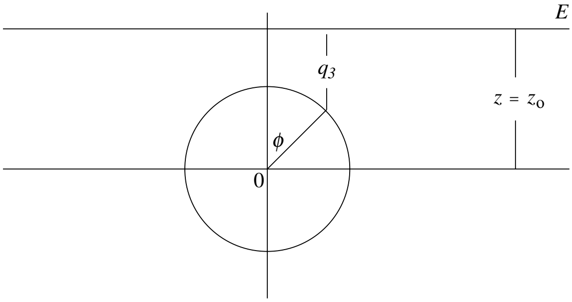

Fig. 1. Relative distance between a plane $E \subset \mathbb{R}_v^3$ and sphere K_{vv_*}.

where $|q_3| = z_\circ - (|v - v_*|/2)\cos\phi$. This will give

$$I \leqslant b_1 |v - v_*|^{\gamma - 2} \left(\frac{\alpha}{\pi}\right)^{1/2} \varphi(v)\varphi(v_*) 2\pi \frac{|v - v_*|^2}{4} \int_0^\pi e^{-\alpha q_3^2} \sin\phi \, d\phi,$$

where the integral has two possible estimates

$$\begin{aligned}
\frac{|v - v_*|^2}{4} \int_0^\pi e^{-\alpha q_3^2} \sin\phi \, d\phi &= \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v - v_*|}{2}}^{z_\circ + \frac{|v - v_*|}{2}} e^{-\alpha q_3^2} \, dq_3 \\
&\le \frac{|v - v_*|}{2} \int_{-\infty}^{\infty} e^{-\alpha q_3^2} \, dq_3 \\
&\le C|v - v_*|
\end{aligned}$$

or

$$\begin{aligned}
\frac{|v - v_*|^2}{4} \int_0^\pi e^{-\alpha q_3^2} \sin\phi \, d\phi &= \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v - v_*}{2}}^{z_\circ + \frac{|v - v_*|}{2}} e^{-\alpha q_3^2} \, dq_3 \\
&\leqslant \frac{|v - v_*|}{2} \int_{z_\circ - \frac{|v - v_*|}{2}}^{z_\circ + \frac{|v - v_*|}{2}} dq_3 \\
&\le C|v - v_*|^2.
\end{aligned}$$

This yields

$$I \leqslant C|v - v_*|^{\gamma - 2} \min\{|v - v_*|, |v - v_*|^2\} \varphi(v)\varphi(v_*).$$

For $\gamma \in [0, 1]$ and positive constants C, C_1

$$C|v - v_*|^{\gamma - 2} \min\{|v - v_*|, |v - v_*|^2\} \leqslant C_1.$$

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}_v^3} \varphi_\alpha J^+(f, f) \, dv \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(v_*) I \, dv \, dv_* \\
&\leqslant C_1 \int_{\mathbb{R}_v^3} \varphi(v) \|f(x, v)\|_{L^\infty} \, dv \int_{\mathbb{R}_v^3} \varphi(v_*) \|f(x, v_*)\|_{L^\infty} \, dv_* \\
&\leqslant C_1 \|f\|^2.
\end{aligned}$$

By taking the limit as $\alpha \to \infty$ the integral on the left hand side of the inequality will concentrate on the plane E, and thus, one establishes the required estimate

$$\sup_E \int_E \varphi(v) \| J^+(f, f) \|_{L^\infty} \, d\sigma(v) \leqslant C_1 \| f \|^2. \quad \blacksquare$$

4. EXISTENCE AND UNIQUENESS

Every set $\mathcal{A}$ is a closed subset of the Banach space $X_\circ$. Hence, we apply the Banach fixed point theorem and achieve existence and uniqueness of solution to problems (2)–(4). To proceed, we assume that f^- (the boundary data) satisfies the following conditions:

$$f^- \geqslant 0, \quad \varphi f^- \in L^1(\mathbb{R}^3_v, L^\infty(\partial\Omega)), \tag{18}$$

$$\sup_\omega \int |\omega - v|^{-1} \varphi(v) \| f^-(., v) \|_{L^\infty(\partial\Omega)} < \infty, \tag{19}$$

$$\sup_E \int_E \varphi f^-(i, v) \, d\sigma(v) < \infty, \quad i = \{0, 1\} \tag{20}$$

$$\inf_{x,v} \nu(f^-) = \nu_\circ > 0. \tag{21}$$

Now, we are in a position to prove existence and uniqueness of a solution to the steady boundary value problems (2)–(4).

Lemma 6. Let $s > 0, r \geqslant 1$. There exists positive constants a_j $(j = 1, \dots, 5)$ such that $A\mathcal{A} \subset \mathcal{A}$ if $\varepsilon < a_5$.

Hence, we need to show that for suitable constants a_j $(j = 1, \dots, 5)$ and $f \in \mathcal{A}$,

$$(a)\ Af \geqslant 0, \qquad (b)\ \| Af \| \leqslant a_1, \qquad (c)\ \nu(Af) \geqslant a_2,$$
$$(d)\ \| Af \|_{-1} \leqslant a_3, \qquad (e)\ \| Af \|_2 \leqslant a_4.$$

Proof. To prove (a), we recall that since both $f^- \geqslant 0$ and $\Pi(\chi(v_1)) \geqslant 0$, it is immediate that $Wf^-(x, v) = f^-(\chi(v_1), v)\Pi(\chi(v_1)) \geqslant 0$. Also, $f \geqslant 0$ will imply $J^+(f, f) \geqslant 0$ and therefore $UJ^+(f, f)(x, v) \geqslant 0$. Hence, it follows that

$$Af \geqslant 0 \quad \text{if} \quad f \geqslant 0.$$

To prove (b), we set

$$a_1 = 2[\|Wf^-\| + \|Wf^-\|_{-1} + \|Wf^-\|_2]$$

and show that for $f \in \mathcal{A}$, $\|Af\| \leqslant a_1$. By the triangle inequality we write

$$\|Af\| \leqslant \|Wf^-\| + \|\varepsilon U J^+(f, f)\|,$$

where we see from the definition of a_1, that $\|Wf^-\| \leqslant a_1/2$. Moreover,

$$\begin{aligned}\|\varepsilon U J^+(f, f)\| &= \int_{\mathbb{R}^3_v} \varphi(v)\varepsilon|v_1|^{-1} \left\| \int_{\chi(v_1)}^{x} J^+(f, f)(y, v)\Pi(x, y)\, dy \right\|_{L^\infty} dv \\ &\leqslant \int_{\mathbb{R}^3_v} \varepsilon|v_1|^{-1}\varphi(v) \left\| \int_0^1 J^+(f, f)(y, v)\Pi(x, y)\, dy \right\|_{L^\infty} dv,\end{aligned}$$

where

$$\begin{aligned}\|J^+(f, f)\|_{L^\infty} &= \sup_x J^+(f, f) \\ &\leqslant J^+(\sup_x f, \sup_x f) = \bar{J}^+(f, f).\end{aligned}$$

Now by using the fact that for $f \in \mathcal{A}$, $\nu(f) \geqslant a_2$, we obtain

$$\|\varepsilon J^+(f, f)\| \leqslant a_2 \int_{\mathbb{R}^3_v} \varphi(v)\bar{J}^+(f, f)\|h\|_{L^\infty}\, dv \tag{22}$$

where from the change of variables $\tau = a_2(x - y)$,

$$h = \varepsilon|v_1|^{-1} \int_0^1 \exp\{-\varepsilon v_1^{-1}\tau\}\, d\tau.$$

Our strategy is to estimate the integral over v_1, and bound the integral over the variable v_2, v_3 by the norm $\|.\|_2$. Hence, we break up the integration in (22) corresponding to the v_1 variable into a sum of two integrals. The first integral is an integral over the set $S_1 = \{|v_1| < \varepsilon^{-1}\}$, and the second, is an integral over the set $S_2 = \{|v_1| > \varepsilon^{-1}\}$. The part containing an integration over the set S_1, we will denote by I_1; and the part containing the integration over S_2, we denote by I_2. So, we write

$$\begin{aligned}\int_{\mathbb{R}^3_v} \varphi(v)\bar{J}^+ h\,dv &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_1} \varphi(v)\bar{J}^+ h\,dv_1\,dv_2\,dv_3 \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_2} \varphi(v)\bar{J}^+ h\,dv_1\,dv_2\,dv_3 \\ &= I_1 + I_2. \end{aligned} \tag{23}$$

In the domain, where $|v_1| > \varepsilon^{-1}(|v_1|^{-1} < \varepsilon)$, we have

$$h \leqslant \varepsilon^2 + \mathrm{O}(\varepsilon^3),$$

which yields the estimate

$$\begin{aligned} I_2 &\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_2} \left(\varepsilon^2 + \mathrm{O}(\varepsilon^3)\right)\varphi(v)\|\bar{J}^+\|_{L^\infty}\,dv + \mathrm{O}(\varepsilon^2) \\ &\simeq \varepsilon^2\|\bar{J}^+\|. \end{aligned}$$

Estimating I_1 we have

$$\begin{aligned} I_1 &= \int_{|v_1|<\varepsilon^{-1}} \frac{\varepsilon}{|v_1|}\int_0^1 \exp\{\frac{-\varepsilon\tau}{|v_1|}\}\,d\tau \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(v)\|\bar{J}^+\|_{L^\infty}\,dv_2\,dv_3\right) dv_1 \\ &\leqslant \int_{S_1} \frac{\varepsilon}{|v_1|}\int_0^1 \exp\{\frac{-\varepsilon\tau}{|v_1|}\}\,d\tau \left(\sup_E \int_E \varphi(v)\|\bar{J}^+(f,f)\|_{L^\infty}\,d\sigma(v)\right) \\ &\leqslant \|\bar{J}^+\|_2 \int_{S_1} \frac{\varepsilon}{|v_1|}\int_0^1 \exp\{\frac{-\varepsilon\tau}{|v_1|}\}\,d\tau\,dv_1. \end{aligned}$$

By explicitly evaluating the inner integral in the above estimate, we have

$$\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau\}\,d\tau = \frac{|v_1|}{\varepsilon}(1-e^{-\varepsilon|v_1|^{-1}}),$$

which gives

$$\begin{aligned} I_1 &\leqslant \|\bar{J}^+\|_2\left[\int_{|v_1|<\varepsilon}(1-e^{-\varepsilon|v_1|^{-1}})\,dv_1 + \int_\varepsilon^{1/\varepsilon}(1-e^{-\varepsilon|v_1|^{-1}})\,dv_1\right] \\ &\leqslant \|\bar{J}^+\|_2\left[2\varepsilon + \int_\varepsilon^{1/\varepsilon}(1-e^{-\varepsilon|v_1|^{-1}})\,dv_1\right]. \end{aligned} \tag{24}$$

Finally we need to estimate the integral in the inequality given in (24). By making a change of variables, $z=\varepsilon/|v_1|$, and doing a Taylor expansion about the point $z=0$, we have

$$\int_{\varepsilon}^{1/\varepsilon}(1-e^{-\varepsilon|v_1|^{-1}})\,dv_1=\varepsilon(\ln 1-\ln\varepsilon^2)-\varepsilon\frac{1}{2}(1-\varepsilon^2)+\mathrm{O}(\varepsilon). \tag{25}$$

Therefore, (24) now yields the estimate

$$I_1\leqslant\|\bar{J}^+\|_2[-2\varepsilon\ln\varepsilon+\mathrm{O}(\varepsilon)]$$

and in view of (22) and (23) one gets

$$\begin{aligned}\|\varepsilon UJ^+(f,f)\|&\leqslant a_2(I_1+I_2)\\&\leqslant a_2\|\bar{J}^+\|_2[-2\varepsilon\ln\varepsilon+\mathrm{O}(\varepsilon)]+\varepsilon^2a_2\|\bar{J}^+\|.\end{aligned}$$

Using the results of Lemmas 3 and 4, we have for $f\in\mathcal{A}$

$$\begin{aligned}\|\varepsilon UJ^+(f,f)\|&\leqslant C_1\|f\|^2[-2\varepsilon\ln\varepsilon]+\varepsilon^2C_2\|f\|\|f\|_{-1}+\mathrm{O}(\varepsilon)\|f\|^2\\&\leqslant C_1a_1^2(-2\varepsilon\ln\varepsilon)+\varepsilon^2C_2a_1a_3+\mathrm{O}(\varepsilon)a_1^2,\end{aligned}$$

where, we see that as $\varepsilon\to 0, -2\varepsilon\ln\varepsilon\to 0$. The other terms are even smaller. Hence, we can pick ε small enough, i.e., $\varepsilon<a_5$ such that

$$\|\varepsilon UJ^+(f,f)\|\leqslant\frac{a_1}{2}$$

and

$$\begin{aligned}\|Af\|&\leqslant\|Wf^-\|+\|\varepsilon UJ^+(f,f)\|\\&\leqslant a_1\end{aligned}$$

as required for the proof of (b). In order to prove (c), it suffices to show that $\nu(Wf^-)$ is bounded below by some positive constant. Therefore, we define for any $b>0$ the set

$$I=\{\omega=(\omega_1,\omega_2,\omega_3)\in\mathbb{R}_v^3:b^{-1}<|\omega|<b,\ |\omega_1|>b^{-1}\}.$$

We make the estimate

$$
\begin{aligned}
&\nu(Wf^-)(x,v) \\
&= 2\pi \int_{\mathbb{R}^3_v} |v-w| Wf^-(\chi(w_1),w)\,dw \\
&\geqslant 2\pi \int_I |v-w| f^-(\chi(w_1),w)\Pi(x,\chi(w_1))\,dw \\
&\geqslant 2\pi \int_I |v-w| f^-(\chi(w_1),w)\exp\left\{-\varepsilon w_1^{-1}\int_{\chi(w_1)}^x \nu(z,w)\,dz\right\} dw \\
&= 2\pi \int_I |v-w| f^-(\chi(w_1),w) \\
&\quad\times \exp\left\{-\varepsilon 2\pi w_1^{-1}\int_{\chi(w_1)}^x \left(\int_{\mathbb{R}^3} |w-w'| f(z,w')\,dw'\right) dz\right\} dw,
\end{aligned}
$$

where for $f \in \mathcal{A}$ the integral in the exponential is estimated to be

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |w-w'| f^-(z,w')\,dw' \\
&\leqslant |w|\int_{\mathbb{R}^3} f^-(z,w')\varphi(w')\,dw' + \int_{\mathbb{R}^3} f^-(z,w')\varphi(w')\,dw' \\
&= (1+|w|)\|f\| \\
&\leqslant (1+b)a_1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\nu(Wf^-)(x,v) &\geqslant 2\pi \int_I |v-w| f^-(\chi(w_1),w) e^{-\varepsilon 2\pi b(1+b)a_1}\,dw \\
&= 2\pi e^{-\varepsilon 2\pi b(1+b)a_1}\int_I |v-w| f^-(\chi(w_1),w)\,dw.
\end{aligned}
$$

Let $\delta > 0$, and $0 < \delta \le (1/4)\nu_\circ$ for some positive constant $\nu_\circ$. First choose b such that for all v

$$
\inf_v \int_I |v-w| f^-(\chi(w_1),w)\,dw \geqslant \nu_\circ - \delta \geqslant \frac{3}{4}\nu_\circ,
$$

then choose ε such that

$$
2\pi e^{-\varepsilon 2\pi b(1+b)a_1}(\nu_\circ - \delta) \geqslant \frac{\nu_\circ}{2}.
$$

Letting $a_2 = \nu_\circ/2$, it follows that $\nu(Wf^-) \geqslant a_2$, and hence

$$\nu(Af) \geqslant a_2,$$

which is the desired result. Now, we have to bound $\|Af\|_{-1}$. From the triangle inequality, and the definition of a_1, one has

$$\|Af\|_{-1} \leqslant \frac{a_1}{2} + \|\varepsilon UJ^+(f,f)\|_{-1}.$$

For $f \in \mathcal{A}$, $\nu(f) \geqslant a_2$, and hence

$$\Pi(x,y) \leqslant \exp\{-\varepsilon|v_1|^{-1}a_2(x-y)\}.$$

By applying the definition of the operator $UJ^+(f,f)$ in (7), and the norm $\|.\|_{-1}$, we write

$$\begin{aligned}
&\|UJ^+(f,f)\|_{-1} \\
&\leqslant \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}|v_1|^{-1} \int_0^1 \|J^+(f,f)\|_{L^\infty} \|\Pi(x,y)\|_{L^\infty}\, dy\, dv \\
&\leqslant \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}|v_1|^{-1} \|J^+(f,f)\|_{L^\infty} \left\| \int_0^1 e^{-\varepsilon|v_1|^{-1}a_2\tau}\, d\tau \right\| dv,
\end{aligned}$$

where, we make the change of variables $\tau = x-y$. We explicitly evaluate the integral in the above estimate to be

$$\int_0^1 \exp\{-\varepsilon|v_1|^{-1}\tau a_2\} d\tau \leqslant \frac{|v_1|}{\varepsilon a_2},$$

which then yields

$$\begin{aligned}
\varepsilon\|UJ^+(f,f)\|_{-1} &\leqslant \varepsilon \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v)|v-\omega|^{-1}|v_1|^{-1}\|J^+(f,f)\|_{L^\infty} \left(\frac{|v_1|}{\varepsilon a_2}\right) dv \\
&= a_2^{-1} \int_{\mathbb{R}^3} \varphi(v)|v-\omega|^{-1}\|J^+(f,f)\|_{L^\infty}\, dv \\
&= \|J^+(f,f)\|_{-1} a_2^{-1}.
\end{aligned}$$

Again for $f \in \mathcal{A}$, we have by Lemma 3 that

$$\begin{aligned}
\|\varepsilon UJ^+(f,f)\|_{-1} &\leqslant C\|f\|^2 a_2^{-1} \\
&\leqslant C a_1^2 a_2{}^{-1},
\end{aligned}$$

Thus, by letting $a_3 = a_1/2 + Ca_1^2 a_2{}^{-1}$, we have the required estimate for part (d) of the proof. Finally, the last part of the proof is done in the same manner as before. From the definition of a_1

$$\|Af\|_2 \leqslant \frac{a_1}{2} + \varepsilon\|UJ^+(f,f)\|_2,$$

where by applying the definition of $UJ^+(f,f)$ and the norm $\|.\|_2$, we have the following estimate:

$$\begin{aligned}
&\varepsilon\|UJ^+(f,f)\|_2\\
&\quad\leqslant \varepsilon \sup_E \int_E \varphi(v)\|J^+(f,f)\|_{L^\infty}|v_1|^{-1}\left\|\int_0^1 e^{-\varepsilon|v_1|^{-1}a_2\tau}\,d\tau\right\| dv\\
&\quad\leqslant \sup_E \int_E \|J^+(f,f)\|_{L^\infty} a_2^{-1}\,dv\\
&\quad= a_2^{-1}\|J^+(f,f)\|_2.
\end{aligned}$$

Upon applying Lemma 5, for $f \in \mathcal{A}$ one obtains

$$\begin{aligned}
\|\varepsilon J^+(f,f)\|_2 &\leqslant Ca_2^{-1}\|f\|^2\\
&\leqslant Ca_1^2 a_2^{-1}.
\end{aligned}$$

Letting $a_1/2 + Ca_1^2 a_2^{-1} = a_3$,

$$\|Af\|_2 \leqslant a_3. \quad \blacksquare$$

Proof of Lemma 1. Since $f^-(x,v)$, and $g^-(x,v)$ are prescribed functions at the boundary, it follows that $Wf^- = Wg^-$. Applying the fact that $Af = Wf^- + \varepsilon UJ^+(f,f)$ we have

$$\begin{aligned}
\|Af - Ag\| &= \varepsilon\left\|\big(UJ^+(f,f) - UJ^+(g,g)\big)\right\|\\
&\leqslant \varepsilon \int_{\mathbb{R}^3_v} \varphi(v)|v_1|^{-1}\\
&\qquad \times \left(\int_0^1 \|(J^+(f,f) - J^+(g,g))\|_{L^\infty}\|\Pi(x,y)\|_{L^\infty}\,dy\right) dv.
\end{aligned}$$

From the property of the collision term

$$\|J^+(f,f) - J^+(g,g)\|_{L^\infty} \leqslant \|J^+(f-g,f+g) + J^+(f+g,f-g)\|_{L^\infty}.$$

An application of the symmetry property of $J^+(f, f)$ along with the definition of $\Pi(x, y)$, leads to the estimate

$$\|Af - Ag\| \leqslant 2\varepsilon \int_{\mathbb{R}^3_v} \varphi(v) \|J^+(f-g, f+g)\|_{L^\infty} |v_1|^{-1} \left(\int_0^1 e^{-\varepsilon |v_1|^{-1} \tau} \, d\tau \right) dv, \tag{26}$$

where $\tau = a_2(x-y)$. It is easily verified that the inner integral

$$\int_0^1 e^{-\varepsilon |v_1|^{-1} \tau} \, d\tau = \frac{|v_1|}{\varepsilon} (1 - e^{-\varepsilon |v_1|^{-1}}).$$

Therefore, by decomposing the integration in (26) in the same manner as in Lemma 6, we are able to explicitly evaluate the integration over the v_1 variables, and estimate the part which defines the plane E by the norm $\|.\|_2$. Hence, we write

$$\begin{aligned} \|Af - Ag\| \leqslant\ & 2\varepsilon \int_{|v_1| < \varepsilon^{-1}} |v_1|^{-1} \frac{|v_1|}{\varepsilon} (1 - e^{-\varepsilon |v_1|^{-1}}) dv_1 \\ & \times \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(v) \|J^+(f-g, f+g)\|_{L^\infty} \, dv_2 \, dv_3 \\ & + 2\varepsilon^2 \int_{|v_1| > \varepsilon^{-1}} dv_1 \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(v) \|J^+(f-g, f+g)\|_{L^\infty} \, dv_2 \, dv_3, \end{aligned}$$

where we have used the fact that in the domain, where $|v_1| > \varepsilon^{-1}$

$$\frac{|v_1|}{\varepsilon} (1 - e^{-\varepsilon |v_1|^{-1}}) = 1 + \mathrm{O}(\varepsilon).$$

We can estimate the integral in this domain by the norm $\|.\|$ defined in (10). Hence we have

$$\begin{aligned} \|Af - Ag\| \leqslant\ & 2 \int_{|v_1| < \varepsilon^{-1}} (1 - e^{-\varepsilon |v_1|^{-1}}) \, dv_1 \\ & \times \left\{ \sup_E \int_E \varphi(v) \|J^+(f-g, f+g)\|_{L^\infty} \, d\sigma(v) \right\} \\ & + 2\varepsilon^2 \|J^+(f-g, f+g)\|. \end{aligned}$$

By (25) the integral in the domain, where $|v_1| < \varepsilon^{-1}$ can be approximated to be

$$\begin{aligned}
&\int_{|v_1|<\varepsilon^{-1}} (1-e^{-\varepsilon|v_1|^{-1}})\,dv_1 \\
&= \int_{|v|<\varepsilon} (1-e^{-\varepsilon|v_1|^{-1}})\,dv_1 + \int_{\varepsilon}^{1/\varepsilon} (1-e^{-\varepsilon|v_1|^{-1}})\,dv_1 \\
&\leqslant \int_{|v_1|<\varepsilon} dv_1 + 2\varepsilon \ln\frac{1}{\varepsilon} - \frac{1}{2}\varepsilon + \mathrm{O}(\varepsilon) \\
&\leqslant 2\varepsilon \ln\frac{1}{\varepsilon} + \mathrm{O}(\varepsilon).
\end{aligned}$$

Going back to the original estimate,

$$\begin{aligned}
&\|Af - Ag\| \\
&\quad \leqslant 2\left(\mathrm{O}(\varepsilon) + 2\varepsilon \ln\frac{1}{\varepsilon}\right)\left\{\sup_E \int_E \varphi(v)\|J^+\|_{L^\infty}\,d\sigma(v)\right\} + \nu_\circ \varepsilon^2 \|J^+\| \\
&\quad \leqslant C_1\left[\mathrm{O}(\varepsilon) + \varepsilon \ln\frac{1}{\varepsilon}\right]\|J^+\|_2 + \varepsilon^2\|J^+\|.
\end{aligned}$$

By taking ε to be small, one can ignore the second order epsilon term and focus only on the first term. Now applying Lemma 5 we have for $f, g \in \mathcal{A}$

$$\begin{aligned}
\|Af - Ag\| &\leqslant C_1\left[\varepsilon + \varepsilon \ln\frac{1}{\varepsilon}\right]\|f-g\|\,\|f+g\| \\
&\leqslant C_1\left[\varepsilon + \varepsilon \ln\frac{1}{\varepsilon}\right]\|f-g\|\,(\|f\| + \|g\|) \\
&\leqslant C\left[\varepsilon + \varepsilon \ln\frac{1}{\varepsilon}\right]\|f-g\| \\
&\leqslant C\varepsilon \ln\frac{1}{\varepsilon}\|f-g\|. \quad \blacksquare
\end{aligned}$$

It is immediate that if we choose ε small enough so that $C\varepsilon \ln(1/\varepsilon) < 1$ we have a contraction; and Theorem 1 follows.

5. DIFFUSE REFLECTION

In this section, we extend the results for the inflow case to the case, where we have diffuse boundary conditions. In particular, we study the

Couette problem, and the existence-uniqueness results associated with it. Therefore, we consider the boundary value problem

$$v_1 \frac{\partial f}{\partial x} = \varepsilon J(f, f), \tag{27}$$

$$f(0, v) = M(0, v)N_\circ(f) \quad \text{if } v_1 > 0, \tag{28}$$

$$f(1, v) = M(1, v)N_1(f) \quad \text{if } v_1 < 0, \tag{29}$$

where $N_\circ, N_1$ represent the fluxes entering and exiting the slab, respectively, and are expressed as

$$N_\circ(f) = \int_{v_1<0} |v_1| f(0, v)\, dv,$$

$$N_1(f) = \int_{v_1>0} |v_1| f(1, v)\, dv.$$

The functions $M(i, v)$ are in the one-dimensional case defined to be

$$M(i, v) = (2\pi)^{-1} h_i^2 \exp(-|v|^2 h_i/2),$$

where i belongs to the set $\{0, 1\}$; which are the positions at the two endpoints of the slab. Here, we also recall that h_i are some positive constants determined by the temperature at the two endpoints. By a simple calculation,

$$\int_{v_1>0} M(0, v)|v_1|\, dv = \int_{v_1<0} M(1, v)|v_1|\, dv = 1$$

and therefore upon multiplying $f(i, v)$ by $|v_1|$ and integrating, we obtain

$$\int_{v_1>0} |v_1| f(0, v)\, dv = \int_{v_1>0} |v_1| M(0, v) N_\circ(f)\, dv = N_\circ(f)$$

and

$$\int_{v_1<0} |v_1| f(1, v)\, dv = \int_{v_1<0} |v_1| M(1, v) N_1(f)\, dv = N_1(f).$$

In general, the boundary value problems (25)–(27) does not possess a unique solution. However, we show that for any given constant $C_\circ$ there exists a unique solution to (25)–(27) which satisfies the added constraint

$$N_\circ(f) + N_1(f) = C_\circ. \tag{30}$$

5.1. STRUCTURE OF THE PROBLEM

In order to, find a unique solution for the Couette problem, we seek functions $f^-(i,v)$ in the form

$$f^-(i,v) = M(i,v)N_i,$$

where N_i are constants we need to find. Let V be the solution operator for Eq. (5), and denote by $W_\circ, U_\circ$ the operators (6) and (7) with $\Pi = 1$. Therefore,

$$(W_\circ f^-)(x,v) = f^-(\chi(v_1),v), \tag{31}$$

$$(U_\circ J)(x,v) = v_1^{-1}\int_{\chi(v_1)}^{x} J(f,f)(y,v)\,dy. \tag{32}$$

Integrating Eq. (25) and applying the boundary conditions (26) and (27), the solution to the Couette problem can be expressed in the form

$$f = Vf^-, \quad Vf^- = W_\circ f^- + \varepsilon U_\circ J(Vf^-, Vf^-). \tag{33}$$

Remark 3. We see that Wf^- prescribes the function at the boundary. As a particle of gas begins to emerge, say at the boundary $x = 1$, its total number of collisions from $x = 1$ to 0 would be determined by $\varepsilon UJ(Vf^-, Vf^-)$.

In view of the above remark, if we multiply Eq. (33) by $|v_1|$ and integrating, we obtain

$$\begin{aligned}&\int_{v_1<0} |v_1| f(0,v)\,dv \\ &= \int_{v_1>0} |v_1| f^-(1,v)\,dv + \varepsilon \int_{v_1>0}\int_1^0 J(Vf^-, Vf^-)(y,v)\,dy\,dv,\end{aligned}$$

which give the following relations:

$$N_\circ = N_1(f) - \varepsilon G(N), \qquad N_1(f) = N_\circ + \varepsilon G(N), \tag{34}$$

where

$$G(N) = \int_{v_1>0}\int_0^1 J(Vf^-, Vf^-)dy\,dv \tag{35}$$

and N is the vector $(N_\circ, N_1)$. By substituting (34) into (30), the following conditions are satisfied:

$$N_\circ(f) = \frac{1}{2}(C_\circ - \varepsilon G(N)), \tag{36}$$

$$N_1(f) = \frac{1}{2}(C_\circ + \varepsilon G(N)). \tag{37}$$

In Section 5.2, the contractive property of the operator $\varepsilon G(N)$ is proved. This result makes use of an estimate on the collision gain term, given by ref. 16 and is based on some properties of the collision operator which is a generalization of Carleman's results[10]. We will not prove this in detail here, but for the convenience of the reader, we only present the main idea of the proof.

5.2. Contraction Property of *G*

5.2.1. A Technicality

We consider the space

$$L^\infty_{s,r} = \left\{ f \mid \ \varphi_{sr} |f| \in L^\infty(\mathbb{R}^3_v) \right\}$$

with the weight function $\varphi \equiv \varphi_{sr}$ defined in Section 2, and for $f \in L^\infty_{sr}$ we introduce the norm

$$\|f\|_{sr} = \varphi(v)\|f\|_{L^\infty}.$$

Then,

Lemma 7. If $f \in L^\infty_{s,r}$, $s > 0$, $r > 4$, then $J^+(f, f) \in L^\infty_{s,r}$. Moreover, there exists a positive constant C such that

$$\|J^+(f, f)\|_{s,r} \leqslant C\|f\|^2_{s,r}.$$

Proof. We keep v fixed and making the change of variables

$$(v_*, n) \to (p = v - n(n.(v - v_*)), q = v_* + n(n.(v - v_*))),$$

where it is easy to see that $p - v$ and $q - v$ are orthogonal. Following Carleman,[10] we can derive a representation of $J^+(f, g)$ in which the integration over $dn\, dv_*$ is replaced by an integration over $dq\, dp$, where q

ranges over E_{vp}, the plane, which is orthogonal to $v-p$ and, contains v, and, where p ranges over $\mathbb{R}^3$. This is seen in Fig. 2. Let $p=v-sn$, where s is the scalar quantity $n.(v-v_*)$. For any fixed n and v, one may write $v_*=q-sn$. We can express p in spherical polar coordinates with the origin at v. Hence, we have $p=-sn$ and writing this out in its components yields $(p_1,p_2,p_3)=-s(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ in which we recognize the volume element $dp=s^2\sin\phi\, ds\, d\theta\, d\phi=s^2\, ds\, dn$. Also since dq is the area element of the plane E_{vp}, and $v_*\in\mathbb{R}^3$, we would have $dv_*=dq\, ds$ (see Fig. 2). Hence $dv_*\, dn=dq\, ds\, dn=(1/s^2)dq\, dp$, where $sn=n(n.(v-v_*))=v-p$. Thus $dv_*\, dn=(1/|v-p|^2)\, dq\, dp$ and we can now express $J^+(f,g)$ as

$$J^+(f,g)(v)=2\int_{\mathbb{R}^3_v} f(p)|v-p|^{-2}\left[\int_{E_{vp}} g(q)B\, dq\right]dp, \tag{38}$$

where the factor 2 is due to the fact that each plane is represented by two opposite directions n. The collision kernel in terms of the variables p,q satisfies,

$$B(p-q,n)\leq b_1|v-p|^{\gamma}$$

for some positive constant b_1 and $\gamma\in[0,1]$. Therefore, we have

$$\begin{aligned}|J^+(f,g)| &\leqslant 2b_1\int_{\mathbb{R}^3_v}|f(p)||v-p|^{-2}|v-p|^{\gamma}\left[\int_{E_{vp}}|g(q)|\, dq\right]dp\\ &=2b_1\int_{\mathbb{R}^3_v}|f(p)||v-p|^{-2+\gamma}\left[\int_{E_{vp}}|g(q)|dq\right]dp\\ &=V_{2-\gamma}(|f|G(|g|)),\end{aligned} \tag{39}$$

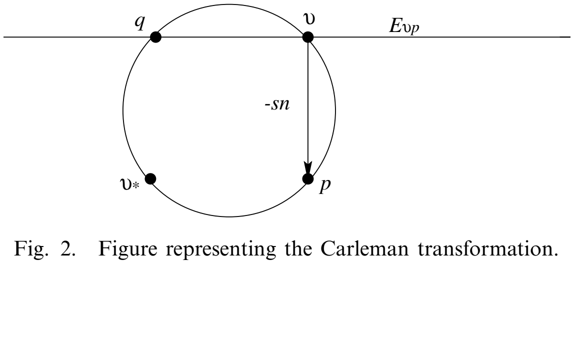

Fig. 2. Figure representing the Carleman transformation.

where we define

$$V_k(f) = 2b_1 \int_{\mathbb{R}_v^3} |v-p|^{-k} f(p)\, dp, \tag{40}$$

$$G(f)(v,p) = \int_{E_{vp}} f(q)\, dq. \tag{41}$$

To proceed further, we define for any fixed v the following sets:

$$D_1 = \left\{ p \in \mathbb{R}^3 \middle| |p| \leq \frac{|v|}{\sqrt{2}} \right\},$$
$$D_2 = \left\{ p \in \mathbb{R}^3 \middle| \frac{|v|}{\sqrt{2}} < |p| < |v| \right\},$$
$$D_3 = \left\{ p \in \mathbb{R}^3 \middle| |p| \geqslant |v| \right\}$$

and we let

$$f_i(p) = f(p)\chi_i(p) \quad \text{for } i = 1, 2, 3,$$

where χ_i is the indicator function of D_i. Making use of the fact, that $J^+(f, f)$ is bilinear with respect to f, and applying the symmetry property of the collision term, we write,

$$\begin{aligned} J^+(f,f) = {} & J^+(f_1,f_1) + 2J^+(f_1,f_2) + 2J^+(f_2,f_3) \\ & + 2J^+(f_1,f_3) + J^+(f_2,f_2) + J^+(f_3,f_3). \end{aligned}$$

We investigate the boundedness of each term separately, using (39) and the definition of $\|.\|_{sr}$. Due to very lengthy calculations, however, we only refer the reader to ref. 13 for details.

Lemma 8. For some positive constant C_1 independent of N, the following estimate is true:

$$|G| \leqslant C_1 \left(\max\{N_\circ, N_1\}\right)^2.$$

Proof. For $f \in \mathcal{A}$ we know that

$$\|f\| \leqslant a_1 = 2\left[\|Wf^-\| + \|Wf^-\|_{-1} + \|Wf^-\|_2\right].$$

We estimate each of the terms above separately, and find an estimate on a_1, in terms of $N_\circ, N_1$. Hence,

$$\begin{aligned}
\|Wf^-\| &= \int_{\mathbb{R}^3_v} \varphi(v) \left\| f^-(i,v)\Pi(i) \right\|_{L^\infty} dv \\
&= \int_{\mathbb{R}^3_v} \varphi(v) \left\| M(i,v) N_i \Pi(i) \right\|_{L^\infty} dv \\
&\leqslant \frac{1}{2\pi} \|h_i^2\|_{L^\infty} \int_{\mathbb{R}^3_v} \varphi(v) e^{-|v|^2 h_i/2} N_i \, dv \\
&= \frac{N_i}{2\pi} \int_{\mathbb{R}^3_v} (1+|v|^2)^r e^{(s-h_i/2)|v|^2} \, dv.
\end{aligned}$$

For $s < \min_i h_i/2$, the integral in the above expression will be bounded by a constant and so

$$\|Wf^-\| \leqslant C_1 N_i \leqslant C_1 \left(\max\{N_\circ, N_1\}\right).$$

Estimating $\|Wf^-\|_{-1}$, we have from the definition of the norm in (9),

$$\begin{aligned}
\|Wf^-\|_{-1} &= \sup_\omega \int_{\mathbb{R}^3_v} \varphi(v) |v-\omega|^{-1} \left\| M(i,v) N_i \right\|_{L^\infty} dv \\
&= \frac{N_i}{2\pi} \|h_i^2\|_{L^\infty} \sup_\omega \int_{\mathbb{R}^3_v} (1+|v|^2)^r e^{(s-h_i)|v|^2/2} |v-\omega|^{-1} \, dv \\
&= C N_i \|e^{-|v|^2 h_i/2}\|_{-1}
\end{aligned}$$

for $s < \min_i \, h_i/2$, and $f(.,v) = e^{-|v|^2 h_i/2}$ (a Maxwellian). Hence,

$$\|Wf^-\|_{-1} = C_2 N_i \left(\max\{N_\circ, N_1\}\right).$$

Similarly, the same estimate for $\|Wf^-\|_2$ follows, namely that:

$$\|Wf^-\|_2 \leqslant C_3 \left(\max\{N_\circ, N_1\}\right).$$

Therefore, it is clear that

$$\|f\| \leqslant a_1 \leqslant C \max\{N_\circ, N_1\} \tag{42}$$

for a positive constant C independent of N, and $f \in \mathcal{A}$. To estimate G, we have

$$
\begin{aligned}
|G| &= \int_{v_1>0}\int_0^1 |J(Vf^-, Vf^-)|\,dy\,dv \\
&\leqslant \int_{v_1>0}\int_0^1 (1+|v|^2)e^{s|v|^2}e^{-s|v|^2}|J^+(Vf^-, Vf^-)|\,dy\,dv \\
&\quad + \int_{v_1>0}\int_0^1 (1+|v|^2)e^{s|v|^2}e^{-s|v|^2}|J^-(Vf^-, Vf^-)|\,dy\,dv.
\end{aligned}
$$

Estimating the part in the above inequality involving the gain term, we have by applying Lemma 7, that for $s < \min_i h_i/2$

$$
\begin{aligned}
\int_{v_1>0}\int_0^1 &(1+|v|^2)e^{s|v|^2}e^{-s|v|^2}|J^+(Vf^-, Vf^-)| \\
&\leqslant \int_{\mathbb{R}^3_v} \|J^+(Vf^-, Vf^-)\|_{s,1} e^{-s|v|^2}\,dv \\
&\leqslant C \int_{\mathbb{R}^3_v} \|f\|_{s,1}^2\,dv \\
&\leqslant C \int_{\mathbb{R}^3_v} \|M(i,v)N\|_{s,r}\,\|f\|_{s,r}\,dv \\
&\leqslant C|N|\,\|f\|.
\end{aligned}
$$

The estimate for the loss term is even simpler and is obtained in the same manner. From inequality (42) we have the required estimate

$$|G| \leqslant C_1\,(\max\{N_\circ, N_1\})^2. \quad \blacksquare$$

Lemma 9. The operator $\varepsilon G(N)$ is a contraction.

We assume two different solutions of the form

$$
\begin{aligned}
f^-(i,v) &= M(i,v)I_i, \\
f^-(i,v) &= M(i,v)J_i,
\end{aligned}
$$

where I_i and J_i are components of I and J, respectively. In Lemma 8, we showed

$$
\begin{aligned}
|G(I)| &\leqslant C_1(\max\{I_\circ, I_1\})^2, \\
|G(J)| &\leqslant C_2(\max\{J_\circ, J_1\})^2
\end{aligned}
$$

for positive constants C_1, and C_2. Thus we have

$$\begin{aligned} G(I-J) &\leqslant C\,(\max\{I_\circ - J_\circ, I_1 - J_1\})^2 \\ &\leqslant C\,(\max(I_\circ - J_\circ, I_1 - J_1))\,(\max(I_\circ + J_\circ, I_1 + J_1)) \\ &\leqslant C|I-J|(I_\circ + I_1 + J_\circ + J_1). \end{aligned}$$

By (30) we know that for any given constants $C_\circ$, C_1 there exists a unique solution satisfying the conditions

$$\begin{aligned} I_\circ + I_1 &= C_\circ, \\ J_\circ + J_1 &= C_1. \end{aligned}$$

Therefore,

$$\varepsilon G(I-J) \leqslant C\varepsilon |I-J|.$$

So, we have that the system in (33) has a unique solution for small ε. Now let $\mathcal{W}^-$ be defined by

$$\mathcal{W}^- = \{f \in L^\infty : ((1+|v|^2)^{-1} Df \in L^\infty\}$$

with Df given by

$$Df = v_1 \frac{\partial f}{\partial x}.$$

We are now ready to state the main result of this section

Theorem 2. There exists a positive constant $\varepsilon_\circ$ such that the problems (27)–(29) has a unique solution in $\mathcal{W}^-$, if $\varepsilon < \varepsilon_\circ$, $s \in (0, \min_i (1/2) h_i)$.

6. CONCLUSION

In this paper, we derived some existence and uniqueness results for the one-dimensional Boltzmann equation in a slab. This relied mainly on the suitable choice of space where we prove the necessary estimates on the collision term. With the use of the norms (9)–(11) it was possible to handle the singularity at $v_1 = 0$, and control the unboundedness in the collision term for hard sphere interactions when $v \to \infty$. As a result, the estimates that were obtained introduced the special properties of the Boltzmann collision operator which eliminated the need for truncating the collision kernel as in ref. 16.

Even though, the results presented here resolve previous difficulties in proving existence and uniqueness for the steady Boltzmann equation, it has its limitations. We notice that these problems were solved with the inclusion of a parameter ε in front of the collision term, which is due to a rescaling of the spatial variables. Our results are obtained by allowing $\varepsilon \to 0$; however, one should point out that in doing this we are not neglecting collisions completely, but we are restricting the problem to particles having large mean free paths.

ACKNOWLEDGMENTS

This work would not have been possible without the helpful guidance of Prof. Reinhard Illner of the University of Victoria, Department of Mathematics and Statistics. I would also like to acknowledge the financial support of NSERC (discovery grant, no. 7847). Prof. Marius Paraschivoiu also deserves my thanks for encouraging me to write this paper.

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